

# The Weyl group of the fine grading of $sl(n, \mathbb{C})$ associated with tensor product of generalized Pauli matrices \*

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January 24, 2011

**Abstract.** We consider the fine grading of  $sl(n, \mathbb{C})$  induced by tensor product of generalized Pauli matrices in the paper. Based on the classification of maximal diagonalizable subgroups of  $PGL(n, \mathbb{C})$  by Havlicek, Patera and Pelantova, we prove that any finite maximal diagonalizable subgroup  $K$  of  $PGL(n, \mathbb{C})$  is a symplectic abelian group and its Weyl group, which describes the symmetry of the fine grading induced by the action of  $K$ , is just the isometry group of the symplectic abelian group  $K$ . For a finite symplectic abelian group, it is also proved that its isometry group is always generated by the transvections contained in it.

## 1 Introduction

The study of gradings of Lie algebras and the symmetries of those gradings is an active research area in recent decades, which are interesting to both mathematicians and physicians. In physics, Lie algebras usually play the role as the algebra of infinitesimal symmetries of a physical system. Knowledge about the gradings of a Lie algebra will greatly help us to understand better the structure of the Lie algebra. Study of the symmetries of those gradings offers a very important tool for describing symmetries in the system of nonlinear equations connected with contraction of a Lie algebra (see e.g. [4]).

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\*Research supported by NSFC Grant No.10801116 and by 'the Fundamental Research Funds for the Central Universities'

Besides the famous Cartan decomposition for semisimple Lie algebras, another well-known example of grading is the grading of  $sl(n, \mathbb{C})$  by the adjoint action of the Pauli group  $\Pi_n$  generated by the  $n \times n$  generalized Pauli matrices, which decomposes  $sl(n, \mathbb{C})$  into direct sum of  $n^2 - 1$  one-dimensional subspaces, each of which consists of semisimple elements.

Let  $L$  be a complex simple Lie algebra. Let  $\text{Aut}(L)$  and  $\text{Int}(L)$  be respectively the automorphism group and inner automorphism group of  $L$ , which are both algebraic groups. A subgroup of  $\text{Aut}(L)$  or  $\text{Int}(L)$  is called *diagonalizable* if it is abelian and consists of semisimple elements. It is not hard to see that there is a natural 1-1 correspondence between gradings of  $L$  and diagonalizable subgroups of  $\text{Aut}(L)$  (see Section 4). A grading is called *inner* if the respective diagonalizable subgroup is in  $\text{Int}(L)$ . A grading (resp. inner grading) of  $L$  is called *fine* if it could not be further refined by any other grading (resp. inner grading). Among the gradings of a Lie algebra, fine (inner) gradings are especially important. It was shown in [9] that the fine gradings of simple Lie algebras correspond to maximal diagonalizable subgroups (which were called MAD-groups in [9]) of  $\text{Aut}(L)$ . Then fine inner grading of simple Lie algebras corresponds to maximal diagonalizable subgroups of  $\text{Int}(L)$ . Given a fine (inner) grading  $\Gamma$ , one can define naturally its Weyl group (see Definition 2.3 of [2]) to describe its symmetry. Assume  $K$  is the maximal diagonalizable subgroup corresponding to  $\Gamma$ , then one can show that its Weyl group is isomorphic to the Weyl group of  $K$ . See Proposition 2.4 and Corollary 2.6 of [2].

After many mathematicians and physicians' contribution, the classification of fine gradings of all the simple Lie algebras are almost done. For example, it can be found in [1] the classification of fine gradings of all the classical simple Lie algebras over an algebraically closed field of characteristic 0. People have also known a lot about the fine gradings for exceptional simple Lie algebras, see [7] for a survey of such results. For the Weyl group of a fine inner grading of a simple Lie algebra  $L$ , if the grading is Cartan decomposition (in which case the corresponding maximal diagonalizable subgroup  $K$  of  $\text{Int}(L)$  is just the maximal torus), then it is well-known that the Weyl group is a finite group generated by reflections; in other cases there is no general result by far. The next step is to study the case  $K$  is discrete, and people have made some explorations in the case  $L = sl(n, \mathbb{C})$ .

Recall that  $PGL(n, \mathbb{C})$  is the inner automorphism group of  $sl(n, \mathbb{C})$ . Let us first review the classification of maximal diagonalizable subgroups of  $PGL(n, \mathbb{C})$ , which correspond to fine inner gradings of  $sl(n, \mathbb{C})$ . Let  $\Pi_n$  be the Pauli group of  $GL(n, \mathbb{C})$  and  $D_n$  be the subgroup of diagonal matrices of  $GL(n, \mathbb{C})$ . Let  $P_n$  and  $D_n$  be the respective images of  $\Pi_n$  and  $D_n$  in

$PGL(n, \mathbb{C})$  under the adjoint action on  $M(n, \mathbb{C})$ . Assume  $n = kl_1 \cdots l_t$  and each  $l_i$  divides  $l_{i-1}$ . The group  $D_k \otimes \Pi_{l_1} \otimes \cdots \otimes \Pi_{l_t}$  consists of all those elements  $A_0 \otimes A_1 \otimes \cdots \otimes A_t$  with  $A_0 \in D_k$  and  $A_i \in \Pi_{l_i}$  for  $1 \leq i \leq t$ . The adjoint action of  $D_k \otimes \Pi_{l_1} \otimes \cdots \otimes \Pi_{l_t}$  on

$$M(k, \mathbb{C}) \otimes M(l_1, \mathbb{C}) \otimes \cdots \otimes M(l_t, \mathbb{C}) \cong M(n, \mathbb{C})$$

induces the embedding

$$D_k \times P_{l_1} \times \cdots \times P_{l_t} \hookrightarrow PGL(n, \mathbb{C}).$$

If we identify  $D_k \times P_{l_1} \times \cdots \times P_{l_t}$  with its image, then it was shown by Havlicek, Patera and Pelantova in Theorem 3.2 of [3] that any maximal diagonalizable subgroup  $K$  of  $PGL(n, \mathbb{C})$  is conjugate to one and only one of the  $D_k \times P_{l_1} \times \cdots \times P_{l_t}$ .

Let  $K$  be a discrete maximal diagonalizable subgroup of  $PGL(n, \mathbb{C})$ . Then  $K \cong P_{l_1} \times P_{l_2} \times \cdots \times P_{l_t}$  where  $n = l_1 l_2 \cdots l_t$  and each  $l_i$  divides  $l_{i-1}$ . Then the fine grading induced by  $K$  also decomposes  $sl(n, \mathbb{C})$  into  $n^2 - 1$  one-dimensional subspaces, each of which consists of semisimple elements. We will show in Section 5 that there is a nonsingular anti-symmetric pairing  $\langle, \rangle$  on  $K$ , such that  $(K, \langle, \rangle)$  is a nonsingular symplectic abelian group (see Definition 2.1). Moreover the pairing  $\langle, \rangle$  is invariant under the Weyl group of  $K$ . It is shown in Proposition 5.6 that there is a one-to-one correspondence between conjugacy classes of finite maximal diagonalizable subgroups of  $PGL(n, \mathbb{C})$  and finite symplectic abelian groups of order  $n^2$ . The following theorem about the structure of the isometry group of a finite nonsingular symplectic abelian group is Theorem 3.16. For the definition of a transvection on a symplectic abelian group, see Definition 3.5.

**Theorem 1.1.** *Let  $(H, \langle, \rangle)$  be a finite nonsingular symplectic abelian group. Then its isometry group is generated by the set of transvections in it.*

If  $K = P_n$ , then in the important paper [4] the authors showed that the respective Weyl group is  $SL(2, \mathbb{Z}_n)$ . If  $m = p^2$  with  $p$  a prime and  $K = P_p \times P_p$ , then in [8] the authors proved that the respective Weyl group is  $Sp(4, \mathbb{Z}_p)$ . Next in [2] we dealt with the case  $n = m^k$  ( $m$  may not be a prime) and  $K = P_m^k$  is the  $k$ -fold direct product of  $P_m$ , and proved that the Weyl group is isomorphic to  $Sp(2k, \mathbb{Z}_n)$  and is generated by transvections. Then, in this paper we deal with the general case that  $K$  is an arbitrary discrete maximal diagonalizable subgroup, and prove the following result in Theorem 6.5 generalizing the previous result.

**Theorem 1.2.** *Let  $K$  be a finite maximal diagonalizable subgroup of  $G = \mathrm{PGL}(n, \mathbb{C})$  and  $W_G(K)$  be its Weyl group. Then  $W_G(K)$  equals the isometry group of  $(K, <, >)$ , and is generated by the set of transvections in it.*

The paper is organized as follows. The definition and classification of finite nonsingular symplectic abelian groups will be reviewed in Section 2. Then in Section 3 we will define transvections on a finite symplectic abelian group and prove Theorem 1.1. Next in section 4 we will review the definitions of the grading of a simple Lie algebra and prove some important properties. Then in Section 5 we will define the anti-symmetric pairing on any finite maximal diagonalizable subgroup of  $\mathrm{PGL}(n, \mathbb{C})$  and prove the 1–1 correspondence between conjugacy classes of finite maximal diagonalizable subgroups of  $\mathrm{PGL}(n, \mathbb{C})$  and nonsingular symplectic abelian groups of order  $n^2$ . In the last section, Theorem 1.2 will be proved.

Finally we introduce some notations in the paper.

For a finite set  $S$ , we will use  $|S|$  to denote its cardinality.

For any  $n \in \mathbb{Z}_+$ , let  $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z} = \{\bar{0}, \bar{1}, \dots, \overline{n-1}\}$ . For simplicity we will just use  $i$  to denote  $\bar{i}$  for  $i = 0, 1, \dots, n-1$ .

Let  $\omega_n = e^{2\pi i/n}$  and  $C_n = \{\omega_n^i | i = 0, 1, \dots, n-1\}$  be the cyclic group of order  $n$  generated by  $\omega_n$ . Sometimes we will identify  $\mathbb{Z}_n$  with  $C_n$  by mapping  $i$  to  $\omega_n^i$ .

#### *Acknowledgments*

The research is supported by NSFC Grant No.10801116 and by 'the Fundamental Research Funds for the Central Universities'. It is finished during the author's visit at the department of mathematics in MIT in 2011. He acknowledges the hospitality of MIT, and would like to take this opportunity to heartily thank David Vogan for drawing his attention to this problem and for Vogan's great generosity in sharing his immense knowledge with him during the research. Proposition 4.3 is due to him.

## **2 Classification of finite nonsingular symplectic abelian groups**

We will follow the definition of symplectic abelian groups in [6], which is defined with respect to any field. But for our purpose we will always assume the field to be  $\mathbb{C}$ , and we will write our abelian groups additively in Section 2 and 3.

Let  $H$  be an abelian group. Recall that an abelian group is automatically a  $\mathbb{Z}$ -module.

**Definition 2.1.** A map

$$\langle, \rangle: H \times H \rightarrow \mathbb{C}^\times$$

is called a pairing of  $H$  into  $\mathbb{C}^\times$  if  $\langle, \rangle$  is  $\mathbb{Z}$ -bilinear. The pairing is called anti-symmetric if for all  $a, b \in H$ ,

$$\langle a, b \rangle = \langle b, a \rangle^{-1}.$$

An anti-symmetric pairing  $\langle, \rangle$  is called nonsingular if  $\langle a, b \rangle = 1$  for any  $b \in H$  implying  $a = 0$ .

**Definition 2.2.** Assume  $\langle, \rangle$  is an anti-symmetric pairing of  $H$  into  $\mathbb{C}^\times$ . Then  $(H, \langle, \rangle)$  is called a symplectic abelian group. A symplectic abelian group  $(H, \langle, \rangle)$  is said to be nonsingular if  $\langle, \rangle$  is nonsingular.

Now assume that  $(H, \langle, \rangle)$  is a nonsingular symplectic abelian group.

A subgroup  $H_0$  of  $H$  is called a nonsingular symplectic abelian subgroup if the restriction  $\langle, \rangle|_{H_0}$  is nonsingular.

Two subgroups  $H_1$  and  $H_2$  of  $H$  are said to be orthogonal, written  $H_1 \perp H_2$ , if  $\langle a, b \rangle = 1$  for any  $a \in H_1, b \in H_2$ .

Two symplectic abelian groups are said to be isometric if there is a group isomorphism between them preserving the respective pairings.

If  $H_1, H_2, \dots, H_n$  is a family of nonsingular symplectic abelian subgroups of  $H$  such that

$$H = H_1 \oplus H_2 \oplus \dots \oplus H_n$$

and

$$H_i \perp H_j, \quad i \neq j,$$

then we will say that  $H$  is the orthogonal direct sum of symplectic abelian subgroups  $H_1, H_2, \dots, H_n$ .

Assume  $n \in \mathbb{Z}^+$  and  $n > 1$ . If a pair of elements  $u, v \in H$  of order  $n$  satisfying  $\langle u, v \rangle = \omega_n$ , then we call  $(u, v)$  a hyperbolic pair of order  $n$  in  $H$ . Let

$$\mathbb{H}_n = \mathbb{Z}_n \times \mathbb{Z}_n \tag{2.1}$$

and  $\langle (i, j), (k, l) \rangle = \omega_n^{il - jk}$  be the pairing on  $\mathbb{H}_n$ , which is clearly nonsingular and anti-symmetric. Then  $(\mathbb{H}_n, \langle, \rangle)$  (or just  $\mathbb{H}_n$ ) is a nonsingular symplectic abelian group, called the hyperbolic group of rank  $n$ . Note that

in [6] the rank  $n$  for a hyperbolic group is assumed to be a power of a prime, but we will not have this restriction in this paper. Let  $u_1 = (1, 0) \in \mathbb{H}_n, v_1 = (0, 1) \in \mathbb{H}_n$ . Then the hyperbolic pair  $(u_1, v_1) \in \mathbb{H}_n^2$  is called the *standard hyperbolic pair* of  $\mathbb{H}_n$ .

**Lemma 2.3.** *Let  $H$  be a finite symplectic abelian group. If  $a, b \in H$  are both of order  $n$  and  $\langle a, b \rangle = \omega_n$ , then  $a$  and  $b$  generate a subgroup  $K$  isometric to  $\mathbb{H}_n$ .*

*Proof.* Assume for some  $i, j \in \mathbb{Z}$ ,  $ia + jb = 0$ . Then  $\langle a, ia + jb \rangle = \omega_n^j = 1$  thus  $n|j$ . Similarly  $n|i$ . So  $K = \{ia + jb | i, j \in \mathbb{Z}_n\}$ . Then

$$K \rightarrow \mathbb{H}_n, ia + jb \mapsto (i, j)$$

is clearly an isometry of symplectic abelian groups.  $\square$

**Lemma 2.4.** *If  $m$  and  $n$  are relatively prime, then  $\mathbb{H}_{mn} \cong \mathbb{H}_m \oplus \mathbb{H}_n$  as symplectic abelian groups.*

*Proof.* Let  $(u_1, v_1) \in \mathbb{H}_m^2$  (resp.  $(u_2, v_2) \in \mathbb{H}_n^2$ ) be the standard hyperbolic pair for  $\mathbb{H}_m$  (resp.  $\mathbb{H}_n$ ). Let

$$a = u_1 + u_2 \in \mathbb{H}_m \oplus \mathbb{H}_n, b = v_1 + v_2 \in \mathbb{H}_m \oplus \mathbb{H}_n.$$

The order of  $a$  and  $b$  are both  $mn$  as  $m$  and  $n$  are relatively prime.

One has  $\langle a, b \rangle = \langle u_1, v_1 \rangle \langle u_2, v_2 \rangle = \omega_m \omega_n = \omega_{mn}^{m+n}$ . As  $mn$  and  $m+n$  are also relatively prime,  $\langle a, ib \rangle = \omega_{mn}$  for some integer  $i$ . Clearly  $\text{ord}(ib)$ , the order of  $ib$ , is still  $mn$ . Then by Lemma 2.3, the subgroup of  $\mathbb{H}_m \oplus \mathbb{H}_n$  generated by  $a$  and  $ib$  is isometric to  $\mathbb{H}_{mn}$ . As  $|\mathbb{H}_{mn}| = |\mathbb{H}_m \oplus \mathbb{H}_n|$ ,

$$\mathbb{H}_m \oplus \mathbb{H}_n \rightarrow \mathbb{H}_{mn}, ja + k(ib) \mapsto (j, k)$$

is an isometry of symplectic abelian groups.  $\square$

**Lemma 2.5.** *Let  $(H, \langle, \rangle)$  be a symplectic abelian group. Let  $a, b \in H$  with  $\text{ord}(a) = i$ ,  $\text{ord}(b) = j$ . Assume  $\langle a, b \rangle = x$ .*

- (1) *If  $l$  is the minimal positive integer such that  $x^l = 1$ , then  $l|i$  and  $l|j$ .*
- (2) *If  $i, j$  are relatively prime then  $x = 1$ .*

*Proof.* (1) As  $x^i = \langle ia, b \rangle = \langle 0, b \rangle = 1$ , one has  $l|i$ . Similarly  $l|j$ .

- (2) Apply (1).  $\square$

For any prime  $p$  dividing  $|H|$ , we will always denote the  $p$ -Sylow subgroup of  $H$  by  $H(p)$ .

**Theorem 2.6.** [Lemma 1.6 and Theorem 1.8 of [6]] Let  $(H, <, >)$  be a finite nonsingular symplectic abelian group. Then

- (1)  $H$  is an orthogonal direct sum of all its Sylow subgroups.
- (2) Assume that  $H(p)$  is a  $p$ -Sylow subgroup of  $H$ . Then

$$H(p) \cong \mathbb{H}_{p^{r_1}} \oplus \mathbb{H}_{p^{r_2}} \oplus \cdots \oplus \mathbb{H}_{p^{r_s}}$$

for some positive integers  $r_1, r_2, \dots, r_s$  with  $r_i \geq r_{i+1}$ .

- (3)  $H$  is an orthogonal direct sum of hyperbolic subgroups  $\mathbb{H}_n$ , where each  $n$  is a power of some prime.

*Proof.* (1) It is proved in Lemma 1.6 of [6]. In fact it is an easy consequence of Lemma 2.5 (2).

- (2) This is proved in Theorem 1.8 of [6] implicitly.

- (3) It is a consequence of (1) and (2). It is also proved in Theorem 1.8 of [6].  $\square$

**Corollary 2.7.** Let  $(H, <, >)$  be a nonidentity finite nonsingular symplectic abelian group. Then there exists positive integers  $l_1, l_2, \dots, l_k$  with each  $l_i | l_{i-1}$  such that

$$H \cong \mathbb{H}_{l_1} \oplus \mathbb{H}_{l_2} \oplus \cdots \oplus \mathbb{H}_{l_k}. \quad (2.2)$$

Such positive integers are uniquely determined by  $H$ .

*Proof.* The existence of such positive integers and the isometry (2.2) follow directly from (1) and (2) of Theorem 2.6. According to the structure theorem of finite abelian groups, such positive integers are uniquely determined by  $H$ .  $\square$

Let  $(H, <, >)$  be a nonsingular symplectic abelian group and  $H_0$  be a finite nonsingular symplectic subgroup. For any  $a \in H$  of order  $n$  and any  $j \in \mathbb{Z}$ , define

$$\omega_n^j \cdot a =_{def} ja. \quad (2.3)$$

Assume that  $H_0 \cong \mathbb{H}_{l_1} \oplus \mathbb{H}_{l_2} \oplus \cdots \oplus \mathbb{H}_{l_k}$  with  $l_i | l_{i-1}$  for all  $i$ . For  $i = 1, \dots, k$  let  $(a_i, b_i)$  be a hyperbolic pair of order  $l_i$  for  $\mathbb{H}_{l_i}$ . Let

$$\pi : H \rightarrow H_0, \quad c \mapsto \sum_{i=1}^k (< c, b_i > \cdot a_i - < c, a_i > \cdot b_i). \quad (2.4)$$

Let  $H_0^\perp = \{a \in H \mid < a, b > = 0, \forall b \in H_0\}$ .

**Proposition 2.8.** (1)  $H_0^\perp$  is a symplectic subgroup of  $H$  and  $H = H_0 \oplus H_0^\perp$  is a direct sum of nonsingular symplectic abelian subgroups.

(2) The map  $\pi$  is independent of the hyperbolic pairs  $(a_i, b_i)$  chosen.

*Proof.* (1) First it is clear that  $H_0^\perp$  is a subgroup of  $H$ .  $H_0 \cap H_0^\perp = 0$  as  $\langle, \rangle|_{H_0}$  is nonsingular. Let  $c \in H$ . Assume  $\langle c, a_i \rangle = \omega_{l_i}^{t_i}$  for  $i = 1, \dots, k$ , then

$$\begin{aligned} \langle \pi(c), a_i \rangle &= \langle -\langle c, a_i \rangle b_i, a_i \rangle \\ &= \langle -t_i b_i, a_i \rangle \\ &= \langle b_i, a_i \rangle^{-t_i} = \omega_{l_i}^{t_i}. \end{aligned}$$

So

$$\langle c, a_i \rangle = \langle \pi(c), a_i \rangle$$

for  $i = 1, \dots, k$ . Similarly

$$\langle c, b_i \rangle = \langle \pi(c), b_i \rangle$$

for  $i = 1, \dots, k$ . Thus one has  $c - \pi(c) \in H_0^\perp$  and

$$c = \pi(c) + (c - \pi(c)) \in H_0 + H_0^\perp. \quad (2.5)$$

So  $H = H_0 \oplus H_0^\perp$ . It is clear that  $\langle, \rangle|_{H_0^\perp}$  must also be nonsingular as  $\langle, \rangle$  is nonsingular. Thus  $H = H_0 \oplus H_0^\perp$  is a direct sum of nonsingular symplectic abelian subgroups.

(2) Let  $\pi' : H \rightarrow H_0$  be defined as in (2.4) with respect to another choice of hyperbolic pairs  $(a'_i, b'_i)$  for each  $\mathbb{H}_{l_i}$ . Then for any  $c \in H$  one also has

$$c = \pi'(c) + (c - \pi'(c)) \in H_0 \oplus H_0^\perp.$$

Comparing to (2.5), as  $H = H_0 \oplus H_0^\perp$  is a direct sum, one must have  $\pi(c) = \pi'(c)$  for any  $c \in H$ .  $\square$

We call the map  $\pi : H \rightarrow H_0$  defined in (2.4) the *projection* of  $H$  onto  $H_0$ .

### 3 Transvections and isometry groups of finite nonsingular symplectic abelian groups

Let  $(H, \langle, \rangle)$  be a finite nonsingular symplectic abelian group.



**Definition 3.1.** Let  $\text{Sp}(H)$  be the set of isometries of  $H$  onto itself. Then  $\text{Sp}(H)$  is clearly a group, called the isometry group of  $H$ .

If  $H = H_1 \oplus H_2$  is a direct sum of nonsingular symplectic abelian subgroups then it is clear that

$$\text{Sp}(H_1) \times \text{Sp}(H_2) \rightarrow \text{Sp}(H), (\phi, \nu)(a, b) = (\phi(a), \nu(b))$$

embeds  $\text{Sp}(H_1) \times \text{Sp}(H_2)$  as a subgroup of  $\text{Sp}(H)$ . For any symplectic subgroup  $H_0$  of  $H$ , as  $H = H_0 \oplus H_0^\perp$ , we will always regard  $\text{Sp}(H_0)$  as a subgroup of  $\text{Sp}(H)$  by the embedding

$$\text{Sp}(H_0) \hookrightarrow \text{Sp}(H_0) \times \text{Sp}(H_0^\perp) \subset \text{Sp}(H), \phi \mapsto (\phi, 1),$$

where 1 denotes the identity map on  $H_0^\perp$ .

**Proposition 3.2.** Let  $H$  be a finite nonsingular symplectic abelian group. Then  $\text{Sp}(H)$  acts transitively on the set of hyperbolic pairs in  $H$  with the same order.

*Proof.* Assume  $(a_1, b_1)$  and  $(a_2, b_2)$  are two hyperbolic pairs in  $H$  with order  $n$ . Let  $H_i = \text{Span}(a_i, b_i)$  for  $i = 1, 2$ . Then  $H_i$  is a symplectic subgroup of  $H$  isometric to  $\mathbb{H}_n$ . Let  $\phi_1 : H_1 \rightarrow H_2$  be the isometry such that  $\phi_1(a_1) = a_2$ ,  $\phi_1(b_1) = b_2$ .

One has  $H = H_1 \oplus H_1^\perp$  and  $H = H_2 \oplus H_2^\perp$ . By Theorem 2.6 (3),  $H_1^\perp \cong H_2^\perp$ . Fix an isometry  $\phi_2 : H_1^\perp \rightarrow H_2^\perp$ . Then

$$\phi = (\phi_1, \phi_2) : H_1 \oplus H_1^\perp \rightarrow H_2 \oplus H_2^\perp, (c, d) \mapsto (\phi_1(c), \phi_2(d))$$

is in  $\text{Sp}(H)$  and maps  $(a_1, b_1)$  to  $(a_2, b_2)$ . Thus  $\text{Sp}(H)$  acts transitively on the set of hyperbolic pairs in  $H$  with the same order.  $\square$

Let

$$\widehat{H} =_{\text{def}} \text{Hom}(H, \mathbb{C}^\times)$$

be the abelian group of characters of  $H$ .

For any  $a \in H$ , define

$$\gamma_a : H \rightarrow \mathbb{C}^\times, b \mapsto \langle a, b \rangle.$$

**Lemma 3.3.** The map

$$\varphi : H \rightarrow \widehat{H}, a \mapsto \gamma_a \tag{3.1}$$

is an isomorphism of abelian groups.

*Proof.* As  $\langle, \rangle$  is  $\mathbb{Z}$ -bilinear,  $\varphi$  is  $\mathbb{Z}$ -linear. As  $\langle, \rangle$  is nonsingular,  $\varphi$  is one-to-one, thus is onto as  $|H| = |\hat{H}|$ .  $\square$

**Definition 3.4.** For any  $\gamma \in \hat{H}$ , let

$$\gamma^* =_{\text{def}} \varphi^{-1}(\gamma).$$

Then for any  $a \in H$ ,  $\gamma(a) = \langle \gamma^*, a \rangle$ . As (3.1) is an isomorphism,  $\gamma$  and  $\gamma^*$  have the same order, and for  $\gamma_1, \gamma_2 \in \hat{H}$ ,

$$(\gamma_1 + \gamma_2)^* = \gamma_1^* + \gamma_2^*. \quad (3.2)$$

Recall that for a vector space  $V$  over  $\mathbb{C}$  (or other field), a linear map  $\phi \in \text{GL}(V)$  is called a transvection if

$$\phi(v) = v + \lambda(v)u$$

for some  $\lambda \in V^*$ ,  $u \in V$  satisfying  $\lambda(u) = 0$ . If  $V$  is a symplectic vector space with  $\langle, \rangle$  the anti-symmetric pairing on it, then any transvection of  $V$  preserving the form  $\langle, \rangle$  must be of the form

$$\phi(v) = v - k \langle u, v \rangle u \quad (3.3)$$

for some  $u \in V$ ,  $k \in \mathbb{C}$ . One knows that  $\text{SL}(V)$  and  $\text{Sp}(V)$  are both generated by their transvections. One can define transvections for a symplectic abelian group analogously.

For any  $b \in H$  with  $b \neq 0$ , assume  $\text{ord}(b) = m$ . For any  $a \in H$ ,  $\langle b, a \rangle = \gamma_b(a)$  takes value in the cyclic group  $C_m = \{\omega_m^i | i = 0, 1, \dots, m-1\}$ . Recall the convention in (2.3).

**Definition 3.5.** For any  $b \in H$  with  $b \neq 0$  and  $k \in \mathbb{Z}$ , define a homomorphism

$$s_{b,k} : H \rightarrow H, \quad a \mapsto a - k(\langle b, a \rangle \cdot b), \quad (3.4)$$

and call it a transvection on  $H$ . Using the identification  $\varphi$  of  $H$  and  $\hat{H}$ , for any  $\gamma \in \hat{H}$  define  $s_{\gamma,k} = s_{\gamma^*,k}$ . Denote  $s_{b,1}$  (resp.  $s_{\gamma,1}$ ) by  $s_b$  (resp.  $s_\gamma$ ) for simplicity.

Then one has

$$s_\gamma(a) = a - \gamma(a)\gamma^*. \quad (3.5)$$

**Lemma 3.6.** (1) For any  $b \in H$  with  $b \neq 0$  and any  $k, j \in \mathbb{Z}$ , one has

$$s_{b,0} = 1, \quad (3.6)$$

$$s_{b,k}s_{b,j} = s_{b,k+j}, \quad (3.7)$$

and

$$s_{b,k}^{-1} = s_{b,-k} \quad (3.8)$$

(2) If  $\text{ord}(b) = m$ , then  $\{s_{b,k} | k \in \mathbb{Z}\}$  is a cyclic group of order  $m$  generated by  $s_{b,1}$ .

*Proof.* (1) (3.6) follows from the definition (3.4). For any  $a \in H$ ,

$$\begin{aligned} s_{b,k}s_{b,j}(a) &= s_{b,k}(a - j(< b, a > \cdot b)) \\ &= (a - j(< b, a > \cdot b)) - k(< b, a - j(< b, a > \cdot b) > \cdot b) \\ &= a - (k + j)(< b, a > \cdot b) \\ &= s_{b,k+j}(a), \end{aligned}$$

So (3.7) holds. Then (3.8) follows from (3.6) and (3.7).

(2) It follows from (1).  $\square$

**Lemma 3.7.** One has  $s_{b,k} \in \text{Sp}(H)$ , where  $b \in H$  with  $b \neq 0$  and  $k \in \mathbb{Z}$ .

*Proof.* By (2) of last lemma one only need to show  $s_b \in \text{Sp}(H)$ . By (3.4) and (3.8) it is clear that  $s_b$  is a  $\mathbb{Z}$ -linear isomorphism of  $H$ , so we only need to prove that  $s_b$  preserves  $<, >$ . Assume  $a, c \in H$  and  $< b, a > = \omega_m^i$ ,  $< b, c > = \omega_m^j$ . Then

$$\begin{aligned} < s_b(a), s_b(c) > &= < a - < b, a > b, c - < b, c > b > \\ &= < a - ib, c - jb > \\ &= < a, c > < b, c >^{-i} < a, b >^{-j} \\ &= < a, c > \omega_m^{j(-i)} \omega_m^{(-i)(-j)} \\ &= < a, c > . \end{aligned}$$

$\square$

Let

$$Q(H) =_{\text{def}} \langle s_{b,k} | 0 \neq b \in H, k \in \mathbb{Z} \rangle \quad (3.9)$$

be the subgroup of  $\text{Sp}(H)$  generated by all the transvections. It is clear that  $Q(H)$  is generated by those  $s_b$  with  $b \in H$  and  $b \neq 0$ . For any element  $b \neq 0$  in a nonsingular symplectic subgroup  $H_0$  of  $H$ ,  $s_b \in Q(H_0)$  can be

regarded as in  $Q(H)$  since  $b \in H$ . Thus  $Q(H_0)$  can be naturally regarded as a subgroup of  $Q(H)$ .

Let  $\text{GL}(2, \mathbb{Z}_n)$  be the group of  $2 \times 2$  invertible matrices in  $M(2, \mathbb{Z}_n)$ . Let

$$\text{SL}(2, \mathbb{Z}_n) = \{A \in \text{GL}(2, \mathbb{Z}_n) \mid \det(A) = 1 \in \mathbb{Z}_n\}.$$

Let

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M(2, \mathbb{Z}_n)$$

and

$$\text{Sp}(2, \mathbb{Z}_n) = \{A \in \text{GL}(2, \mathbb{Z}_n) \mid A^t J A = J\}.$$

It is easily verified that  $\text{SL}(2, \mathbb{Z}_n) = \text{Sp}(2, \mathbb{Z}_n)$ .

**Lemma 3.8.** *One has  $\text{Sp}(\mathbb{H}_n) \cong \text{Sp}(2, \mathbb{Z}_n) = \text{SL}(2, \mathbb{Z}_n)$  and  $\text{Sp}(\mathbb{H}_n) = Q(\mathbb{H}_n)$ . In particular,  $\text{Sp}(\mathbb{H}_n)$  is generated by  $s_{u_1}$  and  $s_{v_1}$ , where  $(u_1, v_1)$  is the standard hyperbolic pair of  $\mathbb{H}_n$ .*

*Proof.* Note that  $(u_1, v_1)$  is an (ordered)  $\mathbb{Z}_n$ -basis for  $\mathbb{H}_n$ . For any  $\varphi \in \text{Sp}(\mathbb{H}_n)$ , one has

$$\varphi(u_1) = a_{11}u_1 + a_{21}v_1, \quad \varphi(v_1) = a_{12}u_1 + a_{22}v_1$$

where  $a_{ij} \in \mathbb{Z}_n$ . Then with respect to the  $\mathbb{Z}_n$ -basis  $(u_1, v_1)$ , the matrix of  $\varphi$  is defined to be

$$C = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad (3.10)$$

which is in  $\text{GL}(2, \mathbb{Z}_n)$ . This defines a map  $\text{Sp}(\mathbb{H}_n) \rightarrow \text{GL}(2, \mathbb{Z}_n)$ , which is an injective homomorphism. For any  $a \in \mathbb{H}_n$ ,  $a = iu_1 + jv_1$  with  $i, j \in \mathbb{Z}_n$ . Then the coordinate  $\tilde{a}$  of  $a$  is denoted  $\tilde{a} = (i, j)^t$ , the transpose of  $(i, j)$ . It is clear that

$$\widetilde{\varphi(a)} = C\tilde{a}. \quad (3.11)$$

If we identify

$$C_n = \{\omega_n^i \mid i = 0, 1, \dots, n-1\} \rightarrow \mathbb{Z}_n, \quad \omega_n^i \rightarrow i,$$

then the matrix of the pairing  $\langle, \rangle$  in the  $\mathbb{Z}_n$ -basis  $(u_1, v_1)$  is  $J$ .

For any  $a, b \in \mathbb{H}_n$ , one has

$$\langle a, b \rangle = \tilde{a}^t J \tilde{b}. \quad (3.12)$$

As  $\varphi \in \text{Sp}(\mathbb{H}_n)$ , by (3.11) and (3.12) one has

$$C^t J C = J,$$

thus  $C \in \text{Sp}(2, \mathbb{Z}_n)$ . So  $\text{Sp}(\mathbb{H}_n) \subset \text{Sp}(2, \mathbb{Z}_n) = \text{SL}(2, \mathbb{Z}_n)$ .

As

$$s_{u_1}(u_1) = u_1, \quad s_{u_1}(v_1) = v_1 - \langle u_1, v_1 \rangle u_1 = v_1 - u_1$$

so the matrix of  $s_{u_1}$  is

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

As

$$s_{v_1}(u_1) = u_1 - \langle v_1, u_1 \rangle v_1 = u_1 + v_1, \quad s_{v_1}(v_1) = v_1$$

so the matrix of  $s_{v_1}$  is

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

It is clear that  $A, B$  generate  $\text{SL}(2, \mathbb{Z}_n)$ , thus  $\text{Sp}(\mathbb{H}_n) = Q(\mathbb{H}_n) = \text{SL}(2, \mathbb{Z}_n)$ .  $\square$

**Corollary 3.9.** *For any  $a \in \mathbb{H}_n$ , if  $\text{ord}(a) = n$  then  $a$  is conjugate to  $u_1$  under  $\text{Sp}(\mathbb{H}_n)$ ; otherwise  $a$  is conjugate to  $lu_1$  for some  $l \in \mathbb{Z}_n$ . In particular all the elements in  $\mathbb{H}_n$  with order  $n$  are conjugate under  $\text{Sp}(\mathbb{H}_n)$ .*

*Proof.* Assume  $a = (i, j)$ . As  $\mathbb{Z}_n$  is a principal ideal ring, the ideal  $I(i, j)$  in  $\mathbb{Z}_n$  generated by  $i, j$  must be generated by some  $l \in \mathbb{Z}_n$ . So  $I(i, j) = I(l)$ .

If  $\text{ord}(a) = n$ ,  $I(l) = \mathbb{Z}_n$ . So there exists  $k, m \in \mathbb{Z}_n$  such that  $ik + jm = 1$ .

Let  $A = \begin{pmatrix} k & m \\ -j & i \end{pmatrix} \in \text{SL}(2, \mathbb{Z}_n)$ . Then  $A(i, j)^t = (1, 0)^t$ , so  $a$  is conjugate to  $u_1$  by  $A$ .

If  $\text{ord}(a) < n$ , then  $(i, j) = l(i', j')$  for some  $a' = (i', j') \in \mathbb{H}_n$  as  $l$  is the greatest common divisor of  $i, j$ . Then  $\text{ord}(a') = n$  and there exists some  $A \in \text{SL}(2, \mathbb{Z}_n)$  such that  $A(a')^t = (1, 0)^t$  and  $Aa^t = (l, 0)^t$ . Thus  $a$  is conjugate to  $lu_1$  by  $A$ .  $\square$

**Lemma 3.10.** *Let  $H = \mathbb{H}_n \oplus \mathbb{H}_n$  and  $\phi : H \rightarrow H$ ,  $(a, b) \mapsto (b, a)$ . Then  $\phi \in Q(H)$ .*

*Proof.* Let  $(u_1, v_1)$  (resp.  $(u_2, v_2)$ ) be the standard hyperbolic pair in  $\mathbb{H}_n \oplus 0$  (resp.  $0 \oplus \mathbb{H}_n$ ). Then  $\phi$  maps  $u_1$  to  $u_2$  and  $v_1$  to  $v_2$ . Let  $x = v_1 + v_2$ , then  $(u_1, x)$  and  $(u_2, x)$  are both hyperbolic pairs of order  $n$ . Let  $H_1 = \text{Span}(u_1, x)$ , then  $H = H_1 \oplus H_1^\perp$ . By Corollary 3.9, there exists

$$\tau = (\tau', 1) \in \text{Sp}(H_1) \times \text{Sp}(H_1^\perp) \subset \text{Sp}(H)$$

such that  $\tau(x) = u_1$ . Similarly there exists  $\varphi \in \text{Sp}(H)$  such that  $\varphi(u_2) = x$ . Then  $\tau\varphi(u_2) = u_1$ . Let  $v_2' = \tau\varphi(v_2)$ .

Assume  $\langle v_2', v_1 \rangle = \omega_n^i$ . Then

$$s_{u_1}^{i-1}(u_1) = u_1,$$

$$s_{u_1}^{i-1}(v_2') = v_2' - (i-1)\langle u_1, v_2' \rangle \cdot u_1 = v_2' - (i-1)u_1.$$

Let  $v_2'' = v_2' - (i-1)u_1$ . Then  $\langle v_2'', v_1 \rangle = \omega_n$ . Let  $q = v_2'' - v_1$ . Note  $\text{ord}(q) = n$ , then

$$s_q(u_1) = u_1 - \langle q, u_1 \rangle \cdot q = u_1,$$

$$s_q(v_2'') = v_2'' - \langle q, v_2'' \rangle \cdot q = v_2'' - \omega_n \cdot q = v_2'' - q = v_1.$$

So the map  $\nu = s_q s_{u_1}^{i-1} \tau \varphi \in Q(H)$  maps  $(u_2, v_2)$  to  $(u_1, v_1)$ . Then  $\nu \phi$  fixes  $(u_1, v_1)$  and maps its orthocomplement  $0 \oplus \mathbb{H}_n$  isometrically onto itself. Then there exists  $\theta = (1, \theta') \in \text{Sp}(\mathbb{H}_n) \times \text{Sp}(\mathbb{H}_n)$  such that  $\theta \nu \phi = 1$ . So  $\phi \in Q(H)$  as  $\theta$  and  $\nu$  are both generated by transvections.  $\square$

**Lemma 3.11.** *Assume that  $H = H(p)$  for some prime  $p$ . Then by Theorem 2.6 (2),  $H = \mathbb{H}_{p^{r_1}} \oplus \mathbb{H}_{p^{r_2}} \oplus \cdots \oplus \mathbb{H}_{p^{r_s}}$  for some positive integers  $r_1, r_2, \dots, r_s$  with  $r_i \geq r_{i+1}$ . For any  $a = (a_1, \dots, a_s) \in H$  with order  $p^{r_1}$ , there exists  $\phi \in Q(H)$  such that  $\phi(a) = b = (b_1, \dots, b_s)$  with  $\text{ord}(b_1) = p^{r_1}$ .*

*Proof.* In this case one has  $\text{ord}(a) = \text{Max}_{i=1}^s \{\text{ord}(a_i)\}$ . If  $\text{ord}(a_1) = p^{r_1}$  then we take  $\phi = 1$ . If  $\text{ord}(a_i) = p^{r_1}$  for some  $i > 1$ , then  $r_i = r_1$ . Then as the subgroups  $\mathbb{H}_{p^{r_1}}$  and  $\mathbb{H}_{p^{r_i}}$  of  $H$  are isometric, by Lemma 3.10 there exists some  $\phi \in Q(\mathbb{H}_{p^{r_1}} \oplus \mathbb{H}_{p^{r_i}}) \subset Q(H)$  such that

$$\phi(a) = \phi(a_1, \dots, a_i, \dots, a_s) = (a_i, \dots, a_1, \dots, a_s) = b.$$

Then  $b_1 = a_i$  and  $\text{ord}(b_1) = p^{r_1}$ .  $\square$

**Lemma 3.12.** *Assume  $H = \mathbb{H}_{l_1} \oplus \mathbb{H}_{l_2} \oplus \cdots \oplus \mathbb{H}_{l_k}$  with  $l_i | l_{i-1}$  for all  $i$ . Then for any  $a = (a_1, \dots, a_k) \in H$  with order  $l_1$ , there exists  $\phi \in Q(H)$  such that  $\phi(a) = b = (b_1, \dots, b_k)$  with  $\text{ord}(b_1) = l_1$ .*

*Proof.* Let  $p_1, \dots, p_s$  be the set of primes dividing  $|H|$ . Then  $H = \oplus_i H(p_i)$  and  $H(p_i) = \mathbb{H}_{l_1}(p_i) \oplus \mathbb{H}_{l_2}(p_i) \oplus \cdots \oplus \mathbb{H}_{l_k}(p_i)$ . Let  $\pi_i : H \rightarrow H(p_i)$  be the projection. Then

$$\pi_i(a) = (a_{1i}, a_{2i}, \dots, a_{ki}) \in \mathbb{H}_{l_1}(p_i) \oplus \mathbb{H}_{l_2}(p_i) \oplus \cdots \oplus \mathbb{H}_{l_k}(p_i).$$

One has  $a = \sum_i \pi_i(a)$ . By last lemma there exists  $\phi_i \in Q(H(p_i)) \subset Q(H)$  such that  $\phi_i(\pi_i(a)) = (b_{1i}, b_{2i}, \dots, b_{ki})$  with  $\text{ord}(b_{1i}) = \text{ord}(\pi_i(a))$ . Let  $\phi = \prod_i \phi_i$ . Then  $\phi(a) = b = (b_1, b_2, \dots, b_k) \in \mathbb{H}_{l_1} \oplus \mathbb{H}_{l_2} \oplus \cdots \oplus \mathbb{H}_{l_k}$ , where  $b_1 = \sum_i b_{1i}$  and  $\text{ord}(b_1) = \text{ord}(b) = \text{ord}(a) = l_1$ .  $\square$

**Lemma 3.13.** *Assume  $H = \mathbb{H}_{l_1} \oplus \mathbb{H}_{l_2} \oplus \cdots \oplus \mathbb{H}_{l_k}$  with  $l_i | l_{i-1}$  for all  $i$ . Then for any  $a = (a_1, a_2, \dots, a_k) \in H$  with order  $l_1$ , there exists  $\phi \in Q(H)$  such that  $\phi(a) = b = (u_1, 0, \dots, 0)$ , where  $u_1 = (1, 0) \in \mathbb{H}_{l_1}$ .*

*Proof.* By last lemma we can assume that  $\text{ord}(a_1) = l_1$ . Then use induction on the number  $t$  of nonzero elements in  $\{a_1, \dots, a_k\}$ . The case  $t = 1$  follows from Corollary 3.9.

Assume there are  $t = l \geq 2$  nonzero elements in  $\{a_1, a_2, \dots, a_k\}$  and the result holds for  $l - 1$ . Without loss of generality we can assume  $a_2 \neq 0$ . As  $\text{ord}(a_1) = l_1$ , there exists some  $b_1 \in \mathbb{H}_{l_1} \subset H$  such that  $\langle b_1, a_1 \rangle = \omega_{l_1}$ . Let  $b = (b_1, a_2, 0, \dots, 0)$ . Then  $\text{ord}(b) = l_1$  and  $\langle b, a \rangle = \omega_{l_1}$ , so

$$\begin{aligned} s_b(a) &= a - \langle b, a \rangle \cdot b \\ &= a - \omega_{l_1} \cdot b = a - b \\ &= (a_1 - b_1, 0, a_3, \dots, a_k). \end{aligned} \tag{3.13}$$

So  $s_b(a) = (a_1 - b_1, 0, a_3, \dots, a_k)$  differs with  $a$  only in the first and second position. As  $s_b(a)$  has order  $l_1$  and has  $l - 1$  nonzero elements, by induction there exists  $\phi_1 \in Q(H)$  such that  $\phi_1(s_b(a)) = (u_1, 0, \dots, 0)$ . Then  $\phi = \phi_1 s_b \in Q(H)$  has the desired property and the result holds for  $t = l$ .  $\square$

**Corollary 3.14.**  *$Q(H)$  acts transitively on the set of elements in  $H$  with maximal order.*

**Lemma 3.15.** *Assume  $H = \mathbb{H}_{l_1} \oplus \mathbb{H}_{l_2} \oplus \cdots \oplus \mathbb{H}_{l_k}$  with  $l_i | l_{i-1}$  for all  $i$  and  $G = \text{Sp}(H)$ . For  $i = 1, \dots, k$  let  $(u_i, v_i)$  be the standard hyperbolic pair in  $\mathbb{H}_{l_i}$ . Then (1)  $G_{u_1} = Q(H)_{u_1}$ . (2)  $G = Q(H)$ .*

*Proof.* We will prove them by induction on  $k$ .

The case  $k = 1$  follows from Lemma 3.8 as we proved there  $\text{Sp}(H) = Q(H)$  if  $H = \mathbb{H}_{l_1}$ . Let  $k > 1$ . Assume (1) and (2) holds for  $k - 1$ .

Assume  $\sigma \in G_{u_1}$ . As  $\langle u_1, v_1 \rangle = \langle \sigma(u_1), \sigma(v_1) \rangle = \langle u_1, \sigma(v_1) \rangle$ ,  $\langle u_1, \sigma(v_1) - v_1 \rangle = 0$  so

$$\sigma(v_1) = v_1 + j_1 u_1 + \sum_{i=2}^k (p_i u_i + q_i v_i)$$

for some  $j_1, p_i, q_i \in \mathbb{Z}$ . By Corollary 3.9 there exists  $\phi_i \in Q(\mathbb{H}_{l_i}) \subset Q(H)$  such that  $\phi_i(p_i u_i + q_i v_i) = j_i u_i$  for  $i \geq 2$ . Let  $\phi = \prod_{i=2}^k \phi_i$ . Then  $\phi \in Q(H)_{u_1}$  and

$$\phi \sigma(v_1) = v_1 + j_1 u_1 + \sum_{i=2}^k j_i u_i.$$

For any  $i$  with  $2 \leq i \leq k$ ,  $s_{u_1+u_i}(u_t) = u_t$  for  $t = 1, \dots, k$ . As  $\text{ord}(u_1 + u_i) = l_1$ ,

$$\begin{aligned} s_{u_1+u_i}(v_1) &= v_1 - \langle u_1 + u_i, v_1 \rangle \cdot (u_1 + u_i) \\ &= v_1 - \omega_{l_1} \cdot (u_1 + u_i) \\ &= v_1 - (u_1 + u_i). \end{aligned} \tag{3.14}$$

Let  $\tau = \prod_{i=2}^k s_{u_1+u_i}^{j_i}$ . Then  $\tau \in Q(H)_{u_1}$  and

$$\tau(\phi\sigma(v_1)) = v_1 + (j_1 - \sum_{i=2}^k j_i)u_1.$$

So  $\tau\phi\sigma$  preserves  $\mathbb{H}_{l_1}$  and also  $\mathbb{H}_{l_1}^\perp = \mathbb{H}_{l_2} \oplus \dots \oplus \mathbb{H}_{l_k}$  thus

$$\tau\phi\sigma \in \text{Sp}(\mathbb{H}_{l_1}) \times \text{Sp}(\mathbb{H}_{l_2} \oplus \dots \oplus \mathbb{H}_{l_k}).$$

By induction

$$\text{Sp}(\mathbb{H}_{l_1}) \times \text{Sp}(\mathbb{H}_{l_2} \oplus \dots \oplus \mathbb{H}_{l_k}) = Q(\mathbb{H}_{l_1}) \times Q(\mathbb{H}_{l_2} \oplus \dots \oplus \mathbb{H}_{l_k}),$$

so

$$\tau\phi\sigma \in Q(\mathbb{H}_{l_1}) \times Q(\mathbb{H}_{l_2} \oplus \dots \oplus \mathbb{H}_{l_k}) \subset Q(H).$$

As  $\tau, \phi \in Q(H)$ , one also has  $\sigma \in Q(H)$ . So  $\sigma \in Q(H)_{u_1}$  then  $G_{u_1} \subset Q(H)_{u_1}$ . As  $G \supset Q(H)$ , one must have  $G_{u_1} = Q(H)_{u_1}$ . So (1) holds for  $k$ .

By Corollary 3.14,  $Q(H)$  acts transitively on the set of elements in  $H$  with maximal order, so does  $G$ . As

$$G_{u_1} = Q(H)_{u_1} \text{ and } |G/G_{u_1}| = |Q(H)/Q(H)_{u_1}|,$$

so  $|G| = |Q(H)|$  thus  $G = Q(H)$ . So (2) also holds for  $k$ .  $\square$

Now we have proved the following theorem.

**Theorem 3.16.** *Let  $H$  be a finite nonsingular symplectic abelian group. Then  $\text{Sp}(H)$  is generated by the set of transvections on  $H$ .*

*Remark 3.17.* If  $H = \mathbb{H}_n^k$ , the  $k$ -fold direct sum of  $\mathbb{H}_n$ , then by choosing some suitable  $\mathbb{Z}_n$ -basis of  $\mathbb{H}_n$ , it is easy to see that

$$\text{Sp}(\mathbb{H}_n^k) \cong \text{Sp}(2k, \mathbb{Z}_n),$$

where

$$\text{Sp}(2k, \mathbb{Z}_n) = \{A \in GL(2k, \mathbb{Z}_n) \mid A^t J_{2k} A = J_{2k}\}$$



with

$$J_{2k} = \begin{pmatrix} 0 & 1 & & \\ -1 & 0 & & \\ & & \ddots & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix} \in M(2k, \mathbb{Z}_n).$$

Thus this theorem implies that in particular  $\mathrm{Sp}(2k, \mathbb{Z}_n)$  is generated by the transvections.

## 4 Fine gradings of Lie algebras and their Weyl groups

4.1. The details of this subsection can be found in Section 4 of [2]. We include it for completeness.

We always assume that  $L$  is a complex simple Lie algebra. Let  $\mathrm{Aut}(L)$  be its automorphism group and  $\mathrm{Int}(L)$  the identity component of  $\mathrm{Aut}(L)$ , called the inner automorphism group of  $L$ . It is clear that  $\mathrm{Aut}(L)$  and  $\mathrm{Int}(L)$  are both algebraic groups.

Let  $\Lambda$  be an additive abelian group. A  $\Lambda$ -grading  $\Gamma$  on  $L$  is the decomposition of  $L$  into direct sum of subspaces

$$\Gamma : L = \bigoplus_{\gamma \in \Lambda} L_\gamma$$

such that

$$[L_\gamma, L_\delta] \subset L_{\gamma+\delta}, \quad \forall \gamma, \delta \in \Lambda.$$

Let  $\Delta = \{\gamma \in \Lambda \mid L_\gamma \neq 0\}$ . We will always assume that  $\Lambda$  is generated by  $\Delta$ , otherwise it could be replaced by its subgroup generated by  $\Delta$ . So  $\Lambda$  is always finitely generated.

Given a  $\Lambda$ -grading  $\Gamma$  on  $L$ , let

$$K = \widehat{\Lambda} =_{\mathrm{def}} \mathrm{Hom}(\Lambda, \mathbb{C}^\times)$$

be the abelian group of characters of  $\Lambda$ . Then  $K$  acts on  $L$  by

$$\sigma \cdot X = \sigma(\gamma)X, \quad \forall X \in L_\gamma, \quad \forall \gamma \in \Lambda, \quad \forall \sigma \in K.$$

This defines an injective homomorphism  $K \rightarrow \mathrm{Aut}(L)$ . So  $K$  can be viewed as a subgroup of  $\mathrm{Aut}(L)$ . Recall that an algebraic group is called *diagonalizable* if it is abelian and consists of semisimple elements. It is easy to see that  $K$  is a diagonalizable algebraic subgroup of  $\mathrm{Aut}(L)$ .

Conversely, given a diagonalizable algebraic subgroup  $K$  of  $\text{Aut}(L)$ , let

$$\Lambda = \widehat{K} =_{\text{def}} \text{Hom}(K, \mathbb{C}^\times)$$

be the (additive) abelian group of homomorphisms from  $K$  to  $\mathbb{C}^\times$  as algebraic groups. Then one has a  $\Lambda$ -grading on  $L$ :

$$\Gamma : L = \bigoplus_{\gamma \in \Lambda} L_\gamma,$$

where  $L_\gamma = \{X \in L \mid \sigma \cdot X = \gamma(\sigma)X, \forall \sigma \in K\}$ . Let

$$\Delta = \Delta(L, K) =_{\text{def}} \{\gamma \in \Lambda \mid L_\gamma \neq 0\}.$$

We call  $\Delta$  the set of *roots* of  $K$  in  $L$ .

Thus there is a natural one-to-one correspondence between gradings of  $L$  by finitely generated abelian groups and diagonalizable algebraic subgroups of  $\text{Aut}(L)$ . A grading is called *inner* if the respective diagonalizable subgroup is in  $\text{Int}(L)$ . A grading (resp. inner grading) of  $L$  is called *fine* if it could not be further refined by any other grading (resp. inner grading). It is clear that the bigger the diagonalizable algebraic subgroup  $K$  is, the finer the corresponding grading is. Thus a grading (resp. inner grading)  $\Gamma$  on  $L$  is fine if and only if the corresponding diagonalizable subgroup  $K$  is a maximal diagonalizable subgroup of  $\text{Aut}(L)$  (resp.  $\text{Int}(L)$ ).

Let  $G$  be either  $\text{Aut}(L)$  or  $\text{Int}(L)$ . Let  $K$  be a maximal diagonalizable subgroup of  $G$ ,  $\Gamma$  the grading on  $L$  induced by the action of  $K$ . One could define the Weyl group  $W_G(\Gamma)$  of the grading  $\Gamma$  with respect to  $G$ , see Definition 2.3 of [2], to describe the symmetry of the grading  $\Gamma$ .

One has the following result in [2].

**Proposition 4.1** (Corollary 2.6 of [2]). *Let  $L$  be a simple Lie algebra and  $G = \text{Int}(L)$ . Let  $K$  be a maximal diagonalizable subgroup of  $G$  and  $\Gamma$  be the corresponding grading on  $L$  induced by the action of  $K$ . Let  $W_G(K) = N_G(K)/K$  be the Weyl group of  $K$  with respect to  $G$ . Then one has  $W_G(\Gamma) = W_G(K)$ .*

4.2. From now on we will always assume  $K \subset G = \text{Int}(L)$  to be a finite maximal diagonalizable subgroup and  $\Delta = \Delta(L, K)$ . Let  $B$  be the Killing form on  $L$ . Recall that a linear subspace  $S$  of  $L$  is called a toral subalgebra if  $[S, S] = 0$  and the endomorphism  $\text{ad}_X$  is semisimple for each  $X \in S$ .

As  $L$  is simple, the adjoint map

$$\text{ad} : L \rightarrow \text{ad}(L), X \mapsto \text{ad}_X$$

is a  $G$ -equivariant isomorphism.

**Definition 4.2.** For any  $\gamma \in \Delta$ , let

$$L_{[\gamma]} =_{\text{def}} \bigoplus_{k \in \mathbb{Z}} L_{k\gamma}.$$

**Proposition 4.3.** (1) One has  $L_0 = 0$ , i.e.,  $0 \notin \Delta$ .

(2) Assume  $\gamma, \delta \in \Delta$  and  $\gamma + \delta \neq 0$ . Then  $B|_{L_\gamma \times L_\delta} = 0$ . For any  $X \in L_\gamma$  with  $X \neq 0$ , there exists  $Y \in L_{-\gamma}$  such that  $B(X, Y) \neq 0$ .

(3) For any  $\gamma \in \Delta$ ,  $L^{\text{Ker } \gamma} = L_{[\gamma]}$  and is a toral subalgebra of  $L$ . One has  $\text{Lie } Z(\text{Ker } \gamma)_0 = \text{ad}(L_{[\gamma]})$  and  $Z(\text{Ker } \gamma)_0$  is an algebraic torus (isomorphic to some  $(\mathbb{C}^\times)^i$ ).

*Proof.* (1) As  $K$  is a maximal diagonalizable subgroup,  $Z_G(K) = K$  by Lemma 2.2 of [2]. As  $K$  is finite,

$$\text{ad}(L^K) = \text{ad}(L)^K = \text{Lie } Z_G(K) = \text{Lie } K = 0.$$

So  $L_0 = L^K = 0$ .

(2) Assume  $\gamma + \delta \neq 0$ . For any  $X \in L_\gamma$  and  $Y \in L_\delta$ ,  $\text{ad}_X \text{ad}_Y$  maps each  $L_\zeta$  into  $L_{\zeta+\gamma+\delta}$  thus  $B(X, Y) = 0$ . Then  $B|_{L_\gamma \times L_\delta} = 0$ . Because  $B$  is nonsingular on  $L$ , the second statement then follows from the first one.

(3) We first prove  $L^{\text{Ker } \gamma} = L_{[\gamma]}$ . Choose some  $\sigma \in K$  satisfying  $\gamma(\sigma) = \omega_m$ , where  $m$  is the order of  $\gamma$ . Then  $\gamma(\sigma)$  is a generator of the cyclic group  $\gamma(K) \cong K/\text{Ker } \gamma$ .  $L^{\text{Ker } \gamma}$  is the direct sum of those  $L_\beta$  with  $\beta$  being identity on  $\text{Ker } \gamma$ . Then  $\beta(\sigma) = \omega_m^k$  for some integer  $k$  as  $\sigma^m \in \text{Ker } \gamma$ . Then  $\beta(\sigma) = \gamma(\sigma)^k = (k\gamma)(\sigma)$ . As  $\text{Ker } \gamma$  and  $\sigma$  generate  $K$ ,  $\beta = k\gamma$ . Thus  $L^{\text{Ker } \gamma} = \bigoplus_k L_{k\gamma} = L_{[\gamma]}$ .

One has  $[L_0, L_{[\gamma]}] = 0$  as  $L_0 = 0$ . Then by Proposition 3.6 of [5],  $L_{[\gamma]}$  is a toral subalgebra of  $L$ . As  $\text{Lie Int}(L) = \text{ad}(L)$ , it is clear that

$$\text{Lie } Z(\text{Ker } \gamma)_0 = \text{Lie } Z(\text{Ker } \gamma) = \text{ad}(L)^{\text{Ker } \gamma} = \text{ad}(L^{\text{Ker } \gamma}) = \text{ad}(L_{[\gamma]}).$$

Thus  $Z(\text{Ker } \gamma)_0$  is an algebraic torus. □

*Remark 4.4.* If  $L$  is a semisimple Lie algebra then all the results in this section still hold.

## 5 Finite maximal diagonalizable subgroups of $\text{PGL}(n, \mathbb{C})$ and anti-symmetric pairings on them

Let  $n \in \mathbb{Z}_+$ . Recall  $\omega_n = e^{2\pi i/n}$ . Let

$$Q_n = \text{diag}(1, \omega_n, \omega_n^2, \dots, \omega_n^{n-1})$$

and

$$P_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Let  $\Pi_n = \{\omega_n^j P_n^k Q_n^l \mid j, k, l = 0, 1, \dots, n-1\}$ . Note  $P_n Q_n = \omega_n Q_n P_n$ . This is a subgroup of  $\mathrm{GL}(n, \mathbb{C})$ , called the *Pauli group* of rank  $n$ .

Let  $D_n$  be the subgroup of diagonal matrices of  $\mathrm{GL}(n, \mathbb{C})$ . Let  $\mathbf{P}_n$  and  $\mathbf{D}_n$  be the respective images of  $\Pi_n$  and  $D_n$  under the adjoint action on  $M(n, \mathbb{C})$ . One knows that

$$\mathbf{P}_n = \{Ad_{P_n}^i Ad_{Q_n}^j \mid i, j = 0, \dots, n-1\} \cong \mathbb{Z}_n \times \mathbb{Z}_n,$$

and that  $\mathbf{D}_n$  and  $\mathbf{P}_n$  are both maximal diagonalizable subgroups of  $\mathrm{PGL}(n, \mathbb{C})$ .

Let  $L = \mathfrak{sl}(n, \mathbb{C})$  and  $G = \mathrm{Int}(L) \cong \mathrm{PGL}(n, \mathbb{C})$ . One has the standard isomorphism

$$M = M(t, \mathbb{C}) \otimes M(l_1, \mathbb{C}) \otimes \cdots \otimes M(l_k, \mathbb{C}) \rightarrow M(n, \mathbb{C}),$$

where  $n = tl_1 \cdots l_k$  and  $l_i \mid l_{i-1}$  for all  $i$ . It induces injective homomorphisms

$$D_t \otimes \Pi_{l_1} \otimes \cdots \otimes \Pi_{l_k} \subset S = \mathrm{GL}(t, \mathbb{C}) \otimes \mathrm{GL}(l_1, \mathbb{C}) \otimes \cdots \otimes \mathrm{GL}(l_k, \mathbb{C}) \rightarrow \mathrm{GL}(n, \mathbb{C}). \quad (5.1)$$

Let  $A = A_0 \otimes A_1 \otimes \cdots \otimes A_k \in S$ . Then for  $X = X_0 \otimes X_1 \otimes \cdots \otimes X_k \in M$ ,

$$\mathrm{Ad}_A(X) = \mathrm{Ad}_{A_0}(X_0) \otimes \mathrm{Ad}_{A_1}(X_1) \otimes \cdots \otimes \mathrm{Ad}_{A_k}(X_k).$$

Thus the adjoint action induces homomorphisms

$$\mathrm{GL}(n, \mathbb{C}) \rightarrow \mathrm{PGL}(n, \mathbb{C}),$$

$$\mathrm{GL}(t, \mathbb{C}) \otimes \mathrm{GL}(l_1, \mathbb{C}) \otimes \cdots \otimes \mathrm{GL}(l_k, \mathbb{C}) \rightarrow \mathrm{PGL}(t, \mathbb{C}) \times \mathrm{PGL}(l_1, \mathbb{C}) \times \cdots \times \mathrm{PGL}(l_k, \mathbb{C}),$$

$$A = A_0 \otimes A_1 \otimes \cdots \otimes A_k \mapsto \mathrm{Ad}_A = (\mathrm{Ad}_{A_0}, \mathrm{Ad}_{A_1}, \dots, \mathrm{Ad}_{A_k}) \quad (5.2)$$

and by restriction

$$D_t \otimes \Pi_{l_1} \otimes \cdots \otimes \Pi_{l_k} \rightarrow \mathbf{D}_t \times \mathbf{P}_{l_1} \times \cdots \times \mathbf{P}_{l_k}.$$

Thus by (5.1) one has injective homomorphisms

$$\mathbf{D}_t \times \mathbf{P}_{l_1} \times \cdots \times \mathbf{P}_{l_k} \subset \mathrm{PGL}(t, \mathbb{C}) \times \mathrm{PGL}(l_1, \mathbb{C}) \times \cdots \times \mathrm{PGL}(l_k, \mathbb{C}) \xrightarrow{\phi} \mathrm{PGL}(n, \mathbb{C}). \quad (5.3)$$

We will identify  $\mathbf{D}_t \times \mathbf{P}_{l_1} \times \cdots \times \mathbf{P}_{l_k}$  with its image in  $\mathrm{PGL}(n, \mathbb{C})$ .

**Theorem 5.1** (Theorem 3.2 of [3]). *Any maximal diagonalizable subgroup of  $\mathrm{PGL}(n, \mathbb{C})$  is conjugate to one and only one of the  $D_t \times P_{l_1} \times \cdots \times P_{l_k}$  with  $n = tl_1 \cdots l_k$  and each  $l_i$  dividing  $l_{i-1}$ .*

**Corollary 5.2.** *Any finite maximal diagonalizable subgroup of  $\mathrm{PGL}(n, \mathbb{C})$  is conjugate to one and only one of the  $P_{l_1} \times \cdots \times P_{l_k}$  with  $n = l_1 \cdots l_k$  and each  $l_i$  dividing  $l_{i-1}$ .*

In the case  $K = P_{l_1} \times \cdots \times P_{l_k}$  is a finite maximal diagonalizable subgroup of  $\mathrm{PGL}(n, \mathbb{C})$ , (5.3) becomes

$$P_{l_1} \times \cdots \times P_{l_k} \subset \mathrm{PGL}(l_1, \mathbb{C}) \times \cdots \times \mathrm{PGL}(l_k, \mathbb{C}) \xrightarrow{\phi} \mathrm{PGL}(n, \mathbb{C}) \quad (5.4)$$

Let  $K \subset \mathrm{PGL}(n, \mathbb{C})$  be a finite maximal diagonalizable subgroup. Let

$$p : \mathrm{GL}(n, \mathbb{C}) \rightarrow \mathrm{PGL}(n, \mathbb{C})$$

be the projection.

**Definition 5.3.** *For any  $\sigma \in K$  fix some  $\tilde{\sigma} \in p^{-1}(\sigma)$ . For any  $\sigma, \tau \in K$ ,  $\tilde{\sigma}\tilde{\tau}\tilde{\sigma}^{-1}\tilde{\tau}^{-1} = lI_n$  as  $p(\tilde{\sigma}\tilde{\tau}\tilde{\sigma}^{-1}\tilde{\tau}^{-1}) = 1$ . Clearly  $l$  is independent of the preimages  $\tilde{\sigma}, \tilde{\tau}$  chosen. Define  $\langle \sigma, \tau \rangle = l$ .*

**Lemma 5.4** (Lemma 3.4 of [2]). *The map  $\langle, \rangle : K \times K \rightarrow \mathbb{C}^\times$  is an anti-symmetric pairing on  $K$ , which is invariant under  $N_G(K)$ .*

**Proposition 5.5.** *Let  $G = \mathrm{PGL}(n, \mathbb{C})$  and  $K$  be a maximal diagonalizable subgroup of  $G$ . If  $K = P_{l_1} \times \cdots \times P_{l_k}$  with  $n = l_1 \cdots l_k$  and each  $l_i$  dividing  $l_{i-1}$ , then  $\langle, \rangle$  is nonsingular on  $K$ . Thus  $(K, \langle, \rangle)$  is a nonsingular symplectic abelian group isometric to  $\mathbb{H}_{l_1} \oplus \cdots \oplus \mathbb{H}_{l_k}$ .*

*Proof.* For  $i = 1, \dots, k$ , let

$$\sigma_i = (1, \dots, 1, Ad_{P_{l_i}}, 1, \dots, 1) \in K, \quad \tau_i = (1, \dots, 1, Ad_{Q_{l_i}}, 1, \dots, 1) \in K$$

where  $Ad_{P_{l_i}}$  and  $Ad_{Q_{l_i}}$  are in the  $i$ -th position. Then  $\{\sigma_i, \tau_i | i = 1, \dots, k\}$  is a set of generators of  $K$  and any element in  $K$  can be written uniquely as  $\sigma_1^{i_1} \tau_1^{j_1} \cdots \sigma_k^{i_k} \tau_k^{j_k}$ . By simple computation one has for  $i \neq j$ ,

$$\langle \sigma_i, \tau_i \rangle = \omega_{l_i}, \quad \langle \sigma_i, \tau_j \rangle = 1,$$

$$\langle \sigma_i, \sigma_j \rangle = 1, \quad \langle \tau_i, \tau_j \rangle = 1.$$

Thus  $(\sigma_i, \tau_i)$  is a hyperbolic pair of order  $l_i$  and spans the symplectic subgroup  $P_{l_i}$  isometric to  $\mathbb{H}_{l_i}$ . It is clear that such subgroups  $P_{l_i}$  are mutually orthogonal to each other. The map

$$P_{l_1} \times \cdots \times P_{l_k} \rightarrow \mathbb{H}_{l_1} \oplus \cdots \oplus \mathbb{H}_{l_k}, \quad \sigma_1^{i_1} \tau_1^{j_1} \cdots \sigma_k^{i_k} \tau_k^{j_k} \mapsto ((i_1, j_1), \dots, (i_k, j_k)) \quad (5.5)$$

is clearly an isometry of nonsingular symplectic abelian groups.  $\square$

As a corollary of Corollary 5.2, Proposition 5.5 and Corollary 2.7 one has the following result.

**Proposition 5.6.** *There is a one-to-one correspondence between conjugacy class of finite maximal diagonalizable subgroups of  $\mathrm{PGL}(n, \mathbb{C})$  and nonsingular symplectic abelian groups of order  $n^2$ .*

## 6 Weyl groups of finite maximal diagonalizable subgroups of $\mathrm{PGL}(n, \mathbb{C})$

Recall that  $L = \mathfrak{sl}(n, \mathbb{C})$  and  $K$  is a finite maximal diagonalizable subgroup of  $G = \mathrm{PGL}(n, \mathbb{C})$ . First we will describe the grading of  $\mathfrak{sl}(n, \mathbb{C})$  and  $\mathfrak{gl}(n, \mathbb{C})$  induced by the action of  $K$ .

At first let  $K = P_n$ . The character group  $\widehat{K}$ , written additively, is generated by  $\beta_n$  and  $\alpha_n$ , which are dual to  $\mathrm{Ad}_{P_n}, \mathrm{Ad}_{Q_n}$ :

$$\beta_n(\mathrm{Ad}_{P_n}) = \omega_n, \quad \beta_n(\mathrm{Ad}_{Q_n}) = 1,$$

$$\alpha_n(\mathrm{Ad}_{P_n}) = 1, \quad \alpha_n(\mathrm{Ad}_{Q_n}) = \omega_n.$$

Thus  $\widehat{K} = \{i\beta_n + j\alpha_n \mid (i, j) \in \mathbb{Z}_n \times \mathbb{Z}_n\} \cong \mathbb{Z}_n^2$ .

One has

$$\beta_n(\mathrm{Ad}_{P_n}^i \mathrm{Ad}_{Q_n}^j) = \omega_n^i = \langle \mathrm{Ad}_{Q_n}^{-1}, \mathrm{Ad}_{P_n}^i \mathrm{Ad}_{Q_n}^j \rangle$$

and

$$\alpha_n(\mathrm{Ad}_{P_n}^i \mathrm{Ad}_{Q_n}^j) = \omega_n^j = \langle \mathrm{Ad}_{P_n}, \mathrm{Ad}_{P_n}^i \mathrm{Ad}_{Q_n}^j \rangle,$$

so

$$\beta_n^* = \mathrm{Ad}_{Q_n}^{-1}, \quad \alpha_n^* = \mathrm{Ad}_{P_n}.$$

As by (3.2) one has  $(\gamma + \delta)^* = \gamma^* \delta^*$ ,

$$(i\beta_n + j\alpha_n)^* = \mathrm{Ad}_{P_n}^j \mathrm{Ad}_{Q_n}^{-i}. \quad (6.1)$$

As

$$\begin{aligned}\mathrm{Ad}_{Q_n}(Q_n^i P_n^j) &= \omega_n^{-j} Q_n^i P_n^j = (i\beta_n - j\alpha_n)(\mathrm{Ad}_{Q_n})Q_n^i P_n^j, \\ \mathrm{Ad}_{P_n}(Q_n^i P_n^j) &= \omega_n^i Q_n^i P_n^j = (i\beta_n - j\alpha_n)(\mathrm{Ad}_{P_n})Q_n^i P_n^j,\end{aligned}$$

one has

$$Q_n^i P_n^j \in L_{i\beta_n - j\alpha_n}. \quad (6.2)$$

In particular

$$P_n \in L_{-\alpha_n}, \quad Q_n \in L_{\beta_n}.$$

Note that  $\mathrm{tr}(Q_n^i P_n^j) = 0$  unless  $(i, j) = (0, 0)$ . Let

$$X_{i\beta_n - j\alpha_n} = Q_n^i P_n^j. \quad (6.3)$$

Then one has the following gradings

$$\begin{aligned}gl(n, \mathbb{C}) &= \oplus_{(i,j)} \mathbb{C}Q_n^i P_n^j = \oplus_{\gamma \in \widehat{K}} \mathbb{C}X_\gamma, \\ sl(n, \mathbb{C}) &= \oplus_{(i,j) \neq (0,0)} \mathbb{C}Q_n^i P_n^j = \oplus_{\gamma \neq 0} \mathbb{C}X_\gamma.\end{aligned}$$

So  $\Delta(gl(n, \mathbb{C}), K) = \widehat{K}$  and  $\Delta(sl(n, \mathbb{C}), K) = \widehat{K} \setminus \{0\}$ . Note that each root space is one-dimensional, and for any  $\gamma \in \widehat{K}$ , by (6.1) and (6.3) one has

$$\gamma^* = (\mathrm{Ad}_{X_\gamma})^{-1}. \quad (6.4)$$

The following result is originally Theorem 10 of [4], for its proof we refer the readers to Proposition 4.4 of [2].

**Theorem 6.1.** *Let  $G = \mathrm{PGL}(n, \mathbb{C})$  and  $K = P_n$ . One has  $W_G(K) \cong \mathrm{SL}(2, \mathbb{Z}_n)$  and is generated by  $s_{\alpha_n}$  and  $s_{\beta_n}$ .*

Next let  $K = P_{l_1} \times \cdots \times P_{l_k}$  with  $n = l_1 \cdots l_k$  and each  $l_i$  dividing  $l_{i-1}$ . As  $M(n, \mathbb{C}) = M(l_1, \mathbb{C}) \otimes \cdots \otimes M(l_k, \mathbb{C})$ , and  $M(l_i, \mathbb{C}) = \oplus_{\gamma \in \widehat{P}_{l_i}} \mathbb{C}X_\gamma$ , one has

$$M(n, \mathbb{C}) = \oplus_{(\gamma_1, \dots, \gamma_k)} \mathbb{C}X_{\gamma_1} \otimes \cdots \otimes X_{\gamma_k}.$$

Note that  $\widehat{K} = \widehat{P}_{l_1} \times \cdots \times \widehat{P}_{l_k}$ . Let  $\gamma = (\gamma_1, \dots, \gamma_k) \in \widehat{K}$  with  $\gamma_i \in \widehat{P}_{l_i}$ . For any  $\sigma = (\sigma_1, \dots, \sigma_k) \in K$ ,  $\gamma(\sigma) = \gamma_1(\sigma_1) \cdots \gamma_k(\sigma_k)$  and one has

$$\begin{aligned}\sigma \cdot X_{\gamma_1} \otimes \cdots \otimes X_{\gamma_k} &= \sigma_1 \cdot X_{\gamma_1} \otimes \cdots \otimes \sigma_k \cdot X_{\gamma_k} \\ &= \gamma_1(\sigma_1)X_{\gamma_1} \otimes \cdots \otimes \gamma_k(\sigma_k)X_{\gamma_k} \\ &= \gamma(\sigma)X_{\gamma_1} \otimes \cdots \otimes X_{\gamma_k}.\end{aligned}$$

So  $X_{\gamma_1} \otimes \cdots \otimes X_{\gamma_k} \in L_\gamma$ . Note that  $\text{tr}(X_{\gamma_1} \otimes \cdots \otimes X_{\gamma_k}) = \prod_i \text{tr}(X_{\gamma_i})$ , which is nonzero if and only if  $\gamma_1 = \cdots = \gamma_k = 0$ . Let

$$Y_\gamma = X_{\gamma_1} \otimes \cdots \otimes X_{\gamma_k}.$$

Then

$$\mathfrak{gl}(n, \mathbb{C}) = \bigoplus_{\gamma \in \widehat{K}} \mathbb{C} Y_\gamma$$

and

$$\mathfrak{sl}(n, \mathbb{C}) = \bigoplus_{\gamma \neq 0} \mathbb{C} Y_\gamma.$$

So  $\Delta(\mathfrak{gl}(n, \mathbb{C}), K) = \widehat{K}$  and  $\Delta(\mathfrak{sl}(n, \mathbb{C}), K) = \widehat{K} \setminus \{0\}$ . Note that each root space is also one-dimensional and consists of semisimple elements.

**Lemma 6.2.** *For any  $\gamma \in \widehat{K}$ ,  $\gamma^* = (\text{Ad}_{Y_\gamma})^{-1}$ .*

*Proof.* For any  $\text{Ad}_X \in K$ , as  $Y_\gamma$  is invertible,

$$Y_\gamma^{-1} X Y_\gamma X^{-1} = Y_\gamma^{-1} (\gamma(\text{Ad}_X) Y_\gamma) = \gamma(\text{Ad}_X) I,$$

so  $\langle (\text{Ad}_{Y_\gamma})^{-1}, \text{Ad}_X \rangle = \gamma(\text{Ad}_X)$ . Thus  $\gamma^* = (\text{Ad}_{Y_\gamma})^{-1}$ .  $\square$

Recall in (5.4) one has the embedding

$$\mathbf{P}_{l_1} \times \cdots \times \mathbf{P}_{l_k} \subset \text{PGL}(l_1, \mathbb{C}) \times \cdots \times \text{PGL}(l_k, \mathbb{C}) \xrightarrow{\phi} \text{PGL}(n, \mathbb{C}).$$

Let  $N(\mathbf{P}_{l_i})$  be the normalizer of  $\mathbf{P}_{l_i}$  in  $\text{PGL}(l_i, \mathbb{C})$ , then clearly  $\phi$  restricts to

$$N(\mathbf{P}_{l_1}) \times \cdots \times N(\mathbf{P}_{l_k}) \xrightarrow{\phi} \text{PGL}(n, \mathbb{C}).$$

The left hand side is in  $N_G(K)$ . As

$$N(\mathbf{P}_{l_i})/\mathbf{P}_{l_i} \cong \text{SL}(2, \mathbb{Z}_{l_i}),$$

one has

$$\text{SL}(2, \mathbb{Z}_{l_1}) \times \cdots \times \text{SL}(2, \mathbb{Z}_{l_k}) \subset W_G(K). \quad (6.5)$$

Let  $\gamma \in \Delta(\mathfrak{sl}(n, \mathbb{C}), K)$  and  $G = \text{PGL}(n, \mathbb{C})$ . Assume the order of  $\gamma$  is  $m$  and choose  $\sigma \in K$  satisfying  $\gamma(\sigma) = \omega_m$ . As  $\sigma \in Z(\text{Ker } \gamma)$ ,  $\text{Ad}_\sigma$  maps  $Z(\text{Ker } \gamma)_0$  into  $Z(\text{Ker } \gamma)_0$ . Let  $f_\sigma : G \rightarrow G$ ,  $\eta \mapsto \sigma \eta \sigma^{-1} \eta^{-1}$ . Then  $f_\sigma(Z(\text{Ker } \gamma)_0) \subset Z(\text{Ker } \gamma)_0$ . Denote  $Z(\text{Ker } \gamma)_0$  by  $Z_0$ .



**Lemma 6.3.** (1) The map  $f_\sigma : Z_0 \rightarrow Z_0$ ,  $\eta \mapsto \sigma\eta\sigma^{-1}\eta^{-1}$  is a continuous epimorphism.

(2) Assume  $\gamma^* \in Z_0$ . Then there exists  $\zeta \in Z_0$  with  $f_\sigma(\zeta) = \gamma^*$ . One has  $\zeta \in N_G(K)$  and  $\text{Ad}_\zeta : K \rightarrow K$ ,  $\tau \mapsto \zeta\tau\zeta^{-1}$  is just the transvection

$$s_\gamma : K \rightarrow K, \tau \mapsto \tau(\gamma^*)^{-\gamma(\tau)}$$

as in (3.5). (Note that as a subgroup of  $G$ ,  $K$  is a multiplicative abelian group.) Thus  $s_\gamma \in W_G(K)$ .

*Proof.* As it was shown in Proposition 4.3 (3) that  $L_{[\gamma]}$  is a toral subalgebra of  $L$ , the lemma follows from Lemma 3.7 of [2].  $\square$

**Lemma 6.4.** For each  $\gamma \in \Delta(\mathfrak{sl}(n, C), K)$ ,  $s_\gamma \in W_G(K)$ .

*Proof.* Assume  $\gamma = (a_1\beta_{l_1} + b_1\alpha_{l_1}, \dots, a_k\beta_{l_k} + b_k\alpha_{l_k})$ , then

$$Y_\gamma = Q_{l_1}^{a_1} P_{l_1}^{-b_1} \otimes \dots \otimes Q_{l_k}^{a_k} P_{l_k}^{-b_k}.$$

By Corollary 3.9 and (6.5) there exists some  $Y_\delta = Q_{l_1}^{c_1} \otimes \dots \otimes Q_{l_k}^{c_k}$  such that  $\text{Ad}_{Y_\gamma}$  is conjugate to  $\text{Ad}_{Y_\delta}$  under  $N_G(K)$ . Assume  $Y_\delta = Q_{l_1}^{c_1} \otimes \dots \otimes Q_{l_k}^{c_k}$  as an element of  $\text{GL}(n, \mathbb{C})$  has order  $m$ . Then  $L_{i\delta} = \mathbb{C}Y_\delta^i$  for  $i = 1, \dots, m-1$  and  $L_{[\delta]} = \bigoplus_{i=1}^{m-1} \mathbb{C}Y_\delta^i$  is an abelian Lie algebra consisting of semisimple elements. We will show  $\text{Ad}_{Y_\delta} \in Z(\text{Ker } \delta)_0$ , then  $\text{Ad}_{Y_\gamma} \in Z(\text{Ker } \gamma)_0$  as  $\text{Ad}_{Y_\gamma}$  and  $\text{Ad}_{Y_\delta}$  are conjugate. Thus  $\gamma^* = (\text{Ad}_{Y_\gamma})^{-1} \in Z(\text{Ker } \gamma)_0$  and  $s_\gamma \in W_G(K)$  by Lemma 6.3.

The set  $D_i$  of eigenvalues of  $Q_{l_i}^{c_i}$  is a cyclic group for each  $i$ . Let  $D$  be the set of eigenvalues of  $Y_\delta$ . For any  $a, b \in D$ ,  $a = a_1 \dots a_k$ ,  $b = b_1 \dots b_k$  with  $a_i, b_i \in D_i$ . Then  $ab^{-1} = (a_1 b_1^{-1}) \dots (a_k b_k^{-1}) \in D$ . So  $D$  is also a group. As the order of  $Y_\delta$  is  $m$ ,  $D$  is a subgroup of the cyclic group  $C_m$ . Then  $D = C_m$  as the order of  $Y_\delta$  is  $m$ .

Let  $\omega = \omega_m$  then in some suitable basis of  $\mathbb{C}^n$ ,

$$Y_\delta = \text{diag}(1, \dots, 1, \omega, \dots, \omega, \dots, \omega^{m-1}, \dots, \omega^{m-1}),$$

where for  $j = 0, 1, \dots, m-1$ , there are  $t_j$  copies of  $\omega^j$  on the diagonal with each  $t_j > 0$ . Let  $s = \frac{2\pi i}{m}$  and

$$A = \text{diag}(0, \dots, 0, s, \dots, s, \dots, (m-1)s, \dots, (m-1)s),$$

where for  $j = 0, 1, \dots, m-1$  there are  $t_j$  copies of  $js$  on the diagonal. Then  $\exp(A) = Y_\delta$ .

Let  $D = (d_{i,j})_{m \times m}$  where  $d_{i,j} = w^{ij}$  for  $i, j = 0, 1, \dots, m-1$ . As  $D$  is invertible, there are unique complex numbers  $c_0, c_1, \dots, c_{m-1}$  satisfying

$$D \cdot (c_0, c_1, \dots, c_{m-1})^t = (0, s, \dots, (m-1)s)^t.$$

Then  $\sum_{i=0}^{m-1} c_i Y_\delta^i = A$  and  $\exp(\sum_{i=0}^{m-1} c_i Y_\delta^i) = Y_\delta$ . Then as

$$\sum_{i=1}^{m-1} c_i Y_\delta^i \in L_{[\delta]} \text{ and } \text{Lie } Z(\text{Ker } \delta)_0 = \text{ad } L_{[\delta]}$$

one has

$$\text{Ad}_{Y_\delta} = \exp(\text{ad}(\sum_{i=1}^{m-1} c_i Y_\delta^i)) \in Z(\text{Ker } \delta)_0.$$

□

Let  $K$  be a finite maximal diagonalizable subgroup of  $G = \text{PGL}(n, \mathbb{C})$ . Recall that  $K$  has a  $W_G(K)$ -invariant anti-symmetric pairing  $\langle, \rangle$  and  $(K, \langle, \rangle)$  is a nonsingular symplectic abelian group by Proposition 5.5. Thus  $W_G(K) \subset \text{Sp}(K)$ , where  $\text{Sp}(K)$  is the isometry group of  $(K, \langle, \rangle)$ .

**Theorem 6.5.** *The Weyl group  $W_G(K)$  equals  $\text{Sp}(K)$ , and is generated by the set of transvections  $s_\gamma$  with  $\gamma \in \Delta(\mathfrak{sl}(n, \mathbb{C}), K)$ .*

*Proof.* By Lemma 6.4 one has  $s_\gamma \in W_G(K)$  for each  $\gamma \in \Delta(\mathfrak{sl}(n, \mathbb{C}), K)$ . As  $\Delta(\mathfrak{sl}(n, \mathbb{C}), K) = \widehat{K} \setminus \{0\}$ ,  $W_G(K)$  contains all  $s_\sigma$  with  $\sigma$  a nonidentity element in  $K$ . By Theorem 3.16 all such  $s_\sigma$  generate  $\text{Sp}(K)$ , thus  $W_G(K) = \text{Sp}(K)$ .

□

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